## General LP Problem

- ***A linear programming problem*** is a minimization or maximization of a linear objective function with $n$ real-valued variables.
- ***The value of the objective function*** for a particular set of values for variables is called its objective value.
- If a particular set of values for variables, satisfies all constraints, it is said to be a ***feasible solution***.
- The set of all feasible solutions is called ***the feasible region***. => ***Interpretation of Geometry***
- A feasible solution that has the minimum (or maximum) objective value is called an ***optimal solution***.
- An optimal solution must satisfy $m$ linear constraints (Inequalities or equalities).
- Strict inequalities are not allowed.

## LP in Standard Form

- ***Maximization*** of a linear function.
- $n$ ***non-negative real-valued variables***.
- $m$ ***linear inequalities*** (<=).

$$
\begin{array}{lll}
max & \sum_{j=1}^{n} c_j x_j & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j \le b_i & for \quad i=1,2,\dots,m \\
 & x_j \ge 0 & for \quad j=1,2,\dots,n
\end{array}
$$

- ***Conversion*** from LP into Standard Form.

$$
\begin{array}{lrcrcl}
min & -2x_1 & + & 3x_2 & & \\
s.t. & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \le & 4 \\
 & & & x_1 & \ge & 0
\end{array}
$$

  - ***Minimization LP is converted to an equivalent maximization*** problem by negating the coefficients of the objective function.

- ***Every variable $x_j$ without the non-negativity constraint is replaced by two non-negative variables*** $x'_j$ and $x''_j$ and each occurence of $x_j$ is replaced by $x'_j - x''_j$.
- ***Each equality constraint is replaced*** by a pair of "opposite" inequality constraints.
- Inequalities are ***"turned around"*** by multiplying both sides by -1.
- ***Renaming*** the variables.

$$
\begin{array}{lcccccccc}
max & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
s.t. & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

**LP in Slack Form**

- Insert ***a slack variable*** into the constraints with inequalities and turn into equalities.

$$
\begin{array}{lll}
max & \sum_{j=1}^{n} c_j x_j & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i & for \quad i = 1, 2, \ldots, m \\
 & x_j \geq 0 & for \quad j = 1, 2, \ldots, n+m
\end{array}
$$

- ***Example***

$$
\begin{array}{lcccccccccccc}
max & 2x_1 & - & 3x_2 & + & 3x_3 & & & & & & & \\
s.t. & x_1 & + & x_2 & - & x_3 & + & x_4 & & & & = & 7 \\
 & -x_1 & - & x_2 & + & x_3 & & & + & x_5 & & = & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & & & & + & x_6 & = & 4
\end{array}
$$

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$$
\begin{array}{ccccccccc}
z & = & 0 & + & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

- Any ***solution of LP in standard form yields a solution of LP in the corresponding slack form*** (with the same objective value) and vice versa.

- Setting right-hand side variables of the ***slack form*** to 0 yields a **basic solution.**
- Left-hand side variables are called **basic**. Right-hand side variables are called **nonbasic**.
- The basic variables are said to constitute a **basis.**
- Note that a basic solution does not need to be feasible.

**SIMPLEX**

1. LP in ***Standard Form***
2. LP in ***Slack Form*** => Basic Solution. If feasible, lucky.
3. ***Pivoting***
    1. Choose the nonbasic variable with positive coefficient and the smallest indices.
    2. Find the binding basic variable.
    3. Selected basic variable entering the nonbasis, the nonbasic variable leaves the nonbasis.
    4. Compute the new objective value.

Terminates at no positive coefficients nonbasic variables or LP is unbound.

**Open Issues**

1. How to decide that LP is feasible?
2. What to do if the initial basic solution is infeasible?
3. How to select entering and leaving variables?
4. How to decide that LP is unbounded?
5. Does SIMPLEX terminate?
6. Does it terminate with an optimal solution?

**Termination**

- SIMPLEX ***computes a feasible basic solution during each iteration***.
- SIMPLEX ***terminate when all coefficients in the objective function are negative or when it becomes obvious that LP is unbounded***.

  Unbound means fix one nonbasic variable and all the basic variables increase, ***no constraint is binding***.
- The ***number of basic solutitons*** is ***finite***:

  Number of basic variables is $m$ and they are selected from among $m + n$ variables. This can be done in

$$\binom{m+n}{m} = \frac{(n+m)!}{n!m!}$$

ways.

- ***Each basic solution has exactly one objective value***. If the objective value increases at each iteration, we will eventually end up with a solution where the coefficients of the objective function are all negative (or we will realize that the LP is unbounded).
- It is possible that ***the objective value*** does not change. => Degeneracy
- It is possible that the same basic solution appears more than once. => Cycle

**Cycling**

- This happens ***even if we use specific rules*** for selecting entering and leaving variables at each iteration, such as
    - The entering variable will always be a nonbasic variable with the largest positive coefficient in the *z*-row. => Max Coefficient
    - If two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript is made to leave. => Smallest indice.
- ***Claim***

  If SIMPLEX fails to terminate then it cycles.
- ***Proof***

  Suppose that SIMPLEX does not cycle but it fails to terminate. So it must generate infinite number of different slack forms. However, the number of different bases is finite. If we can show that a slack form for a given basis is unique then we have a contradiction.
- Avoid Cycling
    - ***Perturb input slightly*** so that it is impossible to have two basic solutions with the same objective value. The perturbation must be such that basic variables in the optimal solutions of the original and perturbed problems are the same. => Haven't Understood.
    - Always choose the entering and leaving variables with the ***smallest indicies***.

**Auxiliary LP**

- To ***check whether the original LP is feasible***.
- The ***auxiliary LP is always feasible and bounded***.
- Optimal value of this auxiliary LP will indicate if the original LP is feasible. => If $x_0 = 0$, then the current solution is also the solution to the original LP. Otherwise, infeasible.

- If original LP is feasible, then the slack form of the auxiliary LP will yield a feasible basic solution to the original LP (and the corresponding slack form).
  - L: LP in standard form:

$$\begin{array}{llcll} \max & \sum_{j=1}^{n} c_j x_j & & & \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & \text{for} & i=1,2,\ldots,m \\ & x_j \geq 0 & \text{for} & j=1,2,\ldots,n \end{array}$$

  - $L_{aux}$: Auxiliary LP:

$$\begin{array}{llcll} \max & -x_0 & & & \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i & \text{for} & i=1,2,\ldots,m \\ & x_j \geq 0 & \text{for} & j=0,1,2,\ldots,n \end{array}$$

**Duality**

- To verify whether the optimal solution is correct as the maximum of the LP meets the minimum of the corresponding dual.
- Upper Bounds on Maximization LP
  - Construct a linear combination of the constraints using nonnegative multipliers.

$$\begin{array}{lrcrcrcrcr} \max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\ s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\ & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\ & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

$$y_1(x_1 - x2 - x_3 + 3x_4) + y_2(5x_1 + x_2 + 3x_3 + 8x_4) + y_3(-x_1 + 2x_2 + 3x_3 - 5x_4) \leq y_1 + 55y_2 + 3y_3$$

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3$$

  - Left-hand side will be an upper bound for the LP if the coefficient at each $x_j$ is at least as big as the corresponding coefficient in the

    objective function.

$$y_1 + 5y_2 - y_3 \geq 4 \qquad -y_1 + y_2 + 2y_3 \geq 1 \qquad 3y_1 + 8y_2 - 5y_3 \geq 3 \qquad -y_1 + 3y_2 + 3y_3 \geq 5$$

  - Any set of nonnegative multipliers $y_i$ satisfying these inequalities also satisfy,

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3$$

- Good Upperbound: Minimize right-hand side s.t. constraints. (Dual)

$$
\begin{array}{lcccccccc}
min & y_1 & + & 55y_2 & + & 3y_3 & & \\
s.t. & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
 & -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
 & 3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3 \\
 & y_1, & & y_2, & & y_3 & \geq & 0
\end{array}
$$

- LP in Standard Form and Its Dual

$$
\begin{array}{lcccccccc}
max & 3x_1 & + & x_2 & + & 2x_3 & & \\
s.t. & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

$$
\begin{array}{lcccccccc}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

- General Dual

$$
\begin{array}{lll}
max & \sum_{j=1}^{n} c_j x_j & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & for \quad i=1,2,\dots,m \\
 & x_j \geq 0 & for \quad j=1,2,\dots,n
\end{array}
$$

$$
\begin{array}{lll}
min & \sum_{i=1}^{m} b_i y_i & \\
s.t. & \sum_{i=1}^{m} a_{ij} y_i \geq c_j & for \quad j=1,2,\dots,n \\
 & y_i \geq 0 & for \quad i=1,2,\dots,m
\end{array}
$$

- $x^*$: feasible solution to the primal LP.
- $y^*$: feasible solution to the dual LP.
- ***Claim***

$$\sum_{j=1}^{n} c_j x_j^* \leq \sum_{i=1}^{m} b_i y_i^*$$

- ***Proof***

$$\begin{aligned} \sum_{j=1}^{n} c_j x_j^* &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* && \text{from the dual} \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \\ &\leq \sum_{i=1}^{m} b_i y_i^* && \text{from the primal} \end{aligned}$$

If equals, then $x^*$ and $y^*$ are optimal solutions to the primal and to the dual LPs, respectively.

- ***Verification with Dual***

## Final Feasible Basic Solution and Corresponding Dual Solution

$$\begin{array}{llcrcrcrcr} \max & z & = & 28 & - & x_3/6 & - & x_5/6 & - & 2x_6/3 \\ s.t. & x_4 & = & 18 & - & x_3/2 & + & x_5/2 & + & 0x_6 \\ & x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\ & x_1 & = & 8 & + & x_3/6 & + & x_5/6 & - & x_6/3 \end{array}$$

Basic variables: $x_1=8$, $x_2=4$, $x_4=18$

Objective value $z = 28$

$$y_i = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}$$

$y_1 = 0$ (since $x_4$ is basic), $y_2 = 1/6$, $y_3 = 2/3$

$$\begin{array}{lrcrcrcl} \min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\ s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\ & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\ & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\ & y_1 & , & y_2 & , & y_3 & \geq & 0 \end{array}$$

- ***Practical Implications***
  - If $m >> n$ in the primal, then the number of constraints in the dual will be much smaller than in the primal.
  - Number of pivots in SIMPLEX is usually less than $1.5m$ and only rarely is higher than $3m$.
  - Number of pivots increases very slowly with $n$.
  - Solving dual will in such cases be more efficient.